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The acoustic theory of the interaction of a shock wave, having an exponential relaxation zone, with a porous medium☆

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ABSTRACT

An analytic solution of the problem of the reflection of a modulated pressure pulse in the form of an "instantaneous jump and exponential relaxation" in a liquid from a plane boundary of a porous medium of infinite extent, saturated with the same liquid, is constructed. Using the analytic solutions obtained, a numerical analysis is given of the development of distinctive features of the reflected and transmitted waves depending on the porosity and permeability of the porous medium, and also the length of the pulse signal.

Considerable attention has recently been devoted to the acoustics of porous media, as it applies to problems of geoacoustics and an analysis of the mechanisms for increasing the extraction of petroleum.[1,2] Despite the abundance of publications, theoretical investigations largely reduce to a dispersion analysis or to a numerical solution of the complete system of equations of saturated porous media. Below, for a limiting case, namely, for the case of an incompressible skeleton of a porous medium, analytic solutions are constructed describing the interaction of a wave of finite length in the form of an "instantaneous jump and exponential relaxation" with a porous medium. This problem was solved previously[3] for a wave in the form of a "step". These solutions may be useful, apart from their independent theoretical interest, when analysing the results of numerical integration.

1. Initial assumptions and fundamental equations

Consider the reflection of a plane one-dimensional pressure wave in a linearly compressible liquid from a plane boundary with a porous and permeable medium, saturated with the same liquid. The skeleton of the porous medium will be assumed to be incompressible. We will direct the coordinate axis perpendicular to the wave front, and we will take the plane of the boundary as the origin of coordinates. The equations of motion in the region of the pure liquid ($x<0$) and in the porous medium ($x>0$) can then be written, in the linearized approximation, in the form[4]

$$\frac{1}{C^2}\frac{\partial p}{\partial t}+\rho_0\frac{\partial u}{\partial x}=0,\quad \rho_0\frac{\partial u}{\partial t}+\frac{\partial p}{\partial x}=0\ (x<0)$$

$$\frac{m}{C^2}\frac{\partial p}{\partial t}+\rho_0\frac{\partial u}{\partial x}=0,\quad \rho_0\frac{\partial u}{\partial t}+m\frac{\partial p}{\partial x}=-\frac{m\mu}{k}u\ (x>0) \tag{1.1}$$

Here u is the velocity of the liquid (the rate of seepage in the zone $x>0$), p is the pressure perturbation, ρ_0 is the unperturbed density of liquid, m is the porosity, k is the permeability of the porous medium, C is the velocity of sound in the liquid and μ is the dynamic viscosity of the liquid.

2. Formulation of the problem

We will assume that, in the initial state, the liquid in the porous medium is at rest and the pressure is uniform

$$u=0,\quad p=0\quad (x>0,\ t\le 0)$$

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Suppose a pressure wave is incident from the side of the pure liquid ($x < 0$), and that the leading edge of the wave is an abrupt shock; after this shock the pressure perturbation relaxes exponentially with a characteristic time t_*. For such a wave, the leading edge of which at $t = 0$ reaches the boundary $x = 0$, according to the first group of equations (1.1) the solution when $t < 0$ can be written in the form

$$p^{(0)} = p_a^{(0)}\exp[-(t - x/C)/t_*]H(t - x/C), \quad u^{(0)} = p^{(0)}/(\rho_0 C) \tag{2.1}$$

Here $p_a^{(0)}$ is the amplitude of the jump and $H(t)$ is Heaviside's function.

When the wave front reaches the surface $x = 0$ (at the instant of time $t = 0$), a reflected wave

$$p^{(r)} = p^{(r)}(t + x/C), \quad u^{(r)} = -p^{(r)}(t + x/C)/(\rho_0 C) \tag{2.2}$$

is formed in the pure-liquid zone ($x < 0$). Hence, the wave perturbations in the region of the pure liquid ($x < 0$) when $t > 0$ can be represented in the form

$$p^{(0r)} = p^{(0)} + p^{(r)}, \quad u^{(0r)} = u^{(0)} + u^{(r)} \tag{2.3}$$

In the region occupied by the porous medium ($x > 0$), as follows from the second group of equations in (1.1), the pressure perturbation $p^{(g)}$ and the velocity $u^{(g)}$ are related by the equation

$$u^{(g)}(x, t) = -\frac{m}{\rho_0}\int_0^t \frac{\partial p^{(g)}(x, t')}{\partial x}\exp\left(\frac{t' - t}{t_\upsilon}\right)dt', \quad t_\upsilon = \frac{\rho_0 k}{m\mu} \tag{2.4}$$

When the pressure wave interacts with the boundary we assume that the pressure and the velocity of the liquid are equal on the side of the pure liquid (the left boundary) and on the side of the liquid which is in the porous medium (the right boundary). Then, on the boundary $x = 0$ when $t \geq 0$ we can write the following relations

$$p(t) = p^{(0)}(t) + p^{(r)}(t) = p^{(g)}(0, t), \quad u(t) = u^{(0)}(t) + u^{(r)}(t) = u^{(g)}(0, t)$$

where $p(t)$ and $u(t)$ are the total pressure and velocity perturbations on the boundary. Hence, taking relations (2.1)–(2.4) into account, we obtain

$$2p^{(0)} - p(t) = \rho_0 C u(t), \quad u(tr) = -\frac{m}{\rho_0}\int_0^t \frac{\partial p^{(g)}(0, t')}{\partial x}\exp\left(\frac{t' - t}{t_\upsilon}\right)dt' \tag{2.5}$$

Using relations (2.1), (2.3) and (2.5), we can write solutions for the distributions of the pressure and velocity perturbations in the region of the pure liquid ($x < 0$) when $t > 0$

$$p^{(0r)} = F^+(x, t), \quad \rho_0 C u^{(0r)} = F^-(x, t)$$

$$F^{\pm}(x, t) = p_a^{(0)} e^{-(t - x/C)t_*} H(t - x/C) \pm [(p(t + x/C) - p_a^{(0)} e^{-(t + x/C)/t_*})H(t + x/C)]$$

3. Construction of the solutions using Laplace transformations

It follows from the second group of equations of (1.1) for $p^{(g)}$ that

$$\frac{\partial^2 p^{(g)}}{\partial t^2} + \frac{1}{t_\upsilon}\frac{\partial p^{(g)}}{\partial t} = C^2\frac{\partial^2 p^{(g)}}{\partial x^2}$$

The solution of this equation, which satisfies the initial and boundary conditions

$$p^{(g)} = 0, \quad \frac{\partial p^{(g)}}{\partial t} = 0 \ (x > 0, t = 0), \quad p^{(g)} = p(t) \ (x = 0, t > 0)$$

using a Laplace transformation,[5] can be written in the form

$$p^{(g)}(x, \tau) = \frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty} \tilde{p}(\lambda)e^{\lambda t - k(\lambda)x}d\lambda; \ \tilde{p}(\lambda) = \int_0^\infty p(t')e^{-\lambda t'}dt', \ k(\lambda) = \frac{\sqrt{\lambda t_\upsilon(1 + \lambda t_\upsilon)}}{C t_\upsilon} \tag{3.1}$$

Hence we obtain

$$\frac{\partial p^{(g)}(0, t)}{\partial x} = -\frac{1}{2\pi i}\int_0^\infty\left[\int_{\sigma - i\infty}^{\sigma + i\infty} k(\lambda)e^{\lambda(t - t')}d\lambda\right]p(t')dt'$$

We substitute this into the expression for $u(t)$ (2.5) and we change the order of integration. After reduction we obtain

$$u(t) = \frac{m}{\rho_0 C}\int_0^\infty A(t-t')p(t')dt';\quad A(t-t') = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\varphi(\lambda t_\upsilon)e^{\lambda(t-t')}d\lambda,\quad \varphi(\lambda t_\upsilon) = \sqrt{\frac{\lambda t_\upsilon}{1+\lambda t_\upsilon}} \tag{3.2}$$

We use the obvious equality

$$A(t-t') = -\frac{\partial}{\partial t'}B(t-t')$$

$$B(t-t') = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\varphi(\lambda t_\upsilon)e^{\lambda(t-t')}\frac{d\lambda}{\lambda} = I_0\left(\frac{t-t'}{2t_\upsilon}\right)H(t-t')\exp\left(-\frac{t-t'}{2t_\upsilon}\right) \tag{3.3}$$

(the integral $B(t-t')$ is evaluated using a well-known formula (Ref. 6, Section 90, formula (21)) and $I_0(z)$ is the Bessel function of imaginary argument). It follows from expressions (3.2) and (3.3) that

$$u(t) = \frac{m}{2t_\upsilon\rho_0 C}\int_0^t K(t-t')p(t')dt' + \frac{m}{\rho_0 C}p(t)$$

$$K(t-t') = \left[I_1\left(\frac{t-t'}{2t_\upsilon}\right) - I_0\left(\frac{t-t'}{2t_\upsilon}\right)\right]\exp\left(-\frac{t-t'}{2t_\upsilon}\right)$$

Substituting this expression into the first equation of (2.5) and making the replacement of variable $z=(t-t')/(2t_\nu)$, we will have the following integral equation for $p(t)$

$$2p^{(0)}(t) - (1+m)p(t) = m\int_0^{t/(2t_\upsilon)}(I_1(z) - I_0(z))e^{-z}p(t-2zt_\upsilon)dz \tag{3.4}$$

Using the well-known relations $I_0(0)=1$, $I_1(z)\approx z/2$ when $z\to 0$ from integral equation (3.4) by taking the limit as $t\to 0$ we obtain

$$p(0) = 2p^{(0)}(0)/(1+m) \tag{3.5}$$

Suppose, when $t\to\infty$, the amplitude of the incident wave has a finite limit $p^{(0)}(\infty)$. Then, taking the limit in integral equation (3.4) as $t\to\infty$ and using the properties of the functions $I_0(z)$ and $I_1(z)$ as $z\to\infty$, we have

$$p(\infty) = 2p^{(0)}(\infty) \tag{3.6}$$

Using a Laplace transformation for Eq. (3.4), we obtain

$$(1+m)\tilde{p}(\lambda) = 2\tilde{p}^{(0)} - \frac{m}{2k_\upsilon}\tilde{K}(\lambda)\tilde{p}(\lambda),\quad \tilde{p}(\lambda) = \int_0^\infty p(t)e^{-\lambda t}d\lambda$$

$$\tilde{p}^{(0)} = \int_0^\infty p^{(0)}(t)e^{-\lambda t}d\lambda,\quad \tilde{K}(\lambda) = 2t_\upsilon\int_0^\infty\left(I_0\left(\frac{t}{2t_\upsilon}\right)e^{-t/(2t_\upsilon)}\right)'e^{-\lambda t}dt = 2t_\upsilon(\varphi(\lambda t_\upsilon)-1) \tag{3.7}$$

(we have used a well-known formula (Ref. 6, Appendix 2, formula (35))).

Hence the following equality follows

$$\tilde{p}(\lambda) = 2\tilde{p}^{(0)}(\lambda t_\upsilon)/(1+m\varphi(\lambda t_\upsilon)) \tag{3.8}$$

where $\tilde{p}^{(0)}(\lambda)$ is the Laplace transform of the pressure perturbation at the boundary $x=0$, corresponding to the incident wave. In the case considered, we have for an incident wave of the "instantaneous jump and exponential relaxation" form

$$\tilde{p}^{(0)}(\lambda) = p_a^{(0)}/(\lambda + 1/t_*) \tag{3.9}$$

4. Conversion of the solutions by the contour-integration method

To simplify the calculations we will introduce the following dimensionless variables and parameters

$$\tau = t/t_\upsilon,\quad \Lambda = \lambda t_\upsilon,\quad X = x/(Ct_\upsilon),\quad P = p/p_a^{(0)},\quad \tau_* = t_*/t_\upsilon,\quad \Lambda_* = \tau_*^{-1}$$

and, taking relations (3.8) and (3.9) into account, we will write expression (3.1) for the pressure perturbation distribution in the zone $x > 0$ in the form

$$P^{(g)}(X,\tau) = \frac{1}{\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \Phi(X,\Lambda)d\Lambda, \ \gamma = \sigma t_v > 0, \ \Phi(X,\Lambda) = \frac{\exp(\Lambda\tau - X\sqrt{\Lambda(\Lambda+1)})}{(1+m\varphi(\Lambda))(\Lambda+\Lambda_*)} \tag{4.1}$$

This integral does not allow of a contour integration when $0 < \Lambda_* < 1$, since the point $\Lambda = -\Lambda_*$ is a first-order pole and falls in the interval $(-1, 0)$ between branching points of the integrand. However, the integral exists and is non-zero when $0 \le X \le \tau$. In fact, representing the expression under the exponential sign in the form $\Lambda(\tau - X) + X(\Lambda - \sqrt{\Lambda(\Lambda+1)})$ and using the retardation theorem,[6] we have

$$P^{(g)}(X,\tau) = H(\tau - X)P_2(X,\tau), \quad P_2(X,\tau) = \frac{1}{\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \exp(\Lambda(\tau - X))\Phi(X,\Lambda)d\Lambda$$

We will represent the expression for $P_2(X, \tau)$ in the form

$$P_2(X,\tau) = \exp(-\Lambda_*\tau)P_1(X,\tau)$$

Differentiating the function $P_1(X, \tau)$ with respect to τ, we get rid of the factor $(\Lambda + \Lambda_*)$ in the denominator of the integrand and we finally obtain

$$\frac{\partial P_1(X,\tau)}{\partial\tau} = \exp(\Lambda_*\tau)P_0(X,\tau), \quad P_0(X,\tau) = \frac{1}{\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \frac{\exp(\Lambda\tau - X\sqrt{\Lambda(1+\Lambda)})}{1+m\varphi(\Lambda)}d\Lambda \tag{4.2}$$

Hence it follows that, when using the contour-integration method for $P_0(X, \tau)$, the required function $P^{(g)}(X, \tau)$ can be found in the form

$$P^{(g)}(X,\tau) = \exp(-\Lambda_*\tau)\int_X^\tau \exp(\Lambda_* t)P_0(X,t)dt + c(X)\exp(-\Lambda_*\tau)$$
$$c(X) = P^{(g)}(X,X)\exp(\Lambda_* X) \tag{4.3}$$

i.e., it is necessary to obtain $P^{(g)}(X, X)$.

As a result of standard calculations using well-known formulae (Ref. 6, Section 90, formulae (19) and (21)) we obtain

$$P^{(g)}(X,X) = 2\exp(-X/2)/(1+m) \tag{4.4}$$

We will evaluate the integral $P_0(X, \tau)$, defined by the second formula of (4.2), by the contour-integration method. The condition of Jordan's lemma[5] is satisfied for this integral. Substituting the result and also expression (4.4) into Eq. (4.3), we obtain

$$\frac{p^{(g)}(X,\tau)}{p_a^{(0)}} = H(\tau - X)\left(\frac{2\exp(-\tau\Lambda_*)\exp(-\tau\Lambda_* + X(2\Lambda_* - 1)/2)}{1+m}\right.$$
$$+\frac{2\exp(-\tau\Lambda_*)}{\pi}\int_{-1}^{1}\sqrt{\frac{1-y}{1+y}}\frac{\exp(\tau(2\Lambda_* - 1 - y)/2) - \exp(X(2\Lambda_* - 1 - y)/2)}{2\Lambda_* - 1 - y}$$
$$\left.\times\frac{\sqrt{1-y^2}\sin(X\sqrt{1-y^2}/2) + m(1+y)\cos(X\sqrt{1-y^2}/2)}{1+m^2+(m^2-1)y}dy\right) \tag{4.5}$$

On the basis of this solution, since $p(t) = p^{(g)}(0, t)$, we obtain the pressure variation at the boundary of the porous medium ($x = 0$)

$$\frac{p(\tau)}{p_a^{(0)}} = \frac{2\exp(-\tau\Lambda_*)}{1+m} + \frac{2\exp(-\tau\Lambda_*)m}{\pi}\int_{-1}^{1}\sqrt{\frac{1-y}{1+y}}\frac{(\exp(\tau(2\Lambda_* - 1 - y)/2) - 1)(1+y)}{(2\Lambda_* - 1 - y)(1+m^2+(m^2-1)y)}dy$$

Substituting expression (4.5) into relations (2.4), after reduction we obtain an expression for the velocity of the liquid in the porous medium; however, it has a removable singularity when $\Lambda_* = 1$. After taking the limit as $\Lambda_* \to 1$, the solution takes the form

$$\frac{u^{(g)}(X,\tau)}{u^{(0)}} = \frac{mH(\tau - X)\exp(-\tau)}{\pi}\left[\frac{\pi(X-\tau+2)\exp(X/2)}{1+m}\right.$$

$$+(\tau - X)\int_{-1}^{1}[\alpha(X,-y) - m\alpha(X,y) + (m+1)\beta(X,y)]\gamma(X,y)dy$$

$$\left.+2\int_{-1}^{1}\frac{[\beta(X,y) - m\alpha(X,y)][\gamma(X,y) - \gamma(\tau,y)]}{1-y}dy\right]$$

$$\alpha(X,y) = (1+y)\sin(X\sqrt{1-y^2}/2), \quad \beta(X,y) = \sqrt{1-y^2}\cos(X\sqrt{1-y^2}/2)$$

$$\gamma(X,y) = \frac{\exp(X(1-y)/2)}{1+m^2+(m^2-1)y}$$

5. Results of numerical calculations

The quadrature formula (Ref. 7, p. 38, formula (3.2.6)) is used for the numerical calculations.

In Fig. 1 we show the change in the dimensionless pressure $P = p/p_a^{(0)}$ when $t > 0$ at the boundary of a porous medium ($x = 0$) for different values of the porosity m and the parameter $\tau = t_*/t_\upsilon$, which is equal to the ratio of the length t_* of the incident pulse to the characteristic time t_υ, which is related to the property of the saturated porous medium.

As follows from the graphs, as also from formula (3.5), the value of the dimensionless pressure immediately after the edge of the incident pulse reaches the boundary $x = 0$ increases to a value $P(0) = 2/(1+m)$. The further behaviour of the function $P(\tau)$ depends on t_*. In particular, the value of the function $P(\tau)$ may fall to zero both under purely monotonic conditions and in the mode with an intermediate maximum. The time taken to reach this maximum increases as the length of the incident signal (defined by the value of t_*) increases.

Hence, the form of the $P(\tau)$ curve for a known value of the amplitude $p_a^{(0)}$ and the length t_* of the incident signal gives information both on the value of the porosity m and on the characteristic time t_υ, defined by the permeability and porosity of the skeleton and the properties of the liquid saturating the porous medium. If the properties of the saturating liquid (the viscosity and velocity of sound) are known, the value of t_υ is determined by the porosity and permeability of the porous medium. Consequently, one can judge the value of the porosity and permeability from the form of the $P(\tau)$ curve. Note that the $P(\tau)$ curve corresponds to the oscillogram of the pressure for a sensor placed at the boundary of the porous medium ($x = 0$).

In Fig. 2 we show the evolution of the pressure and velocity fields in the porous medium ($x > 0$) and in the zone of the pure liquid ($x < 0$) for $m = 0{,}1$.

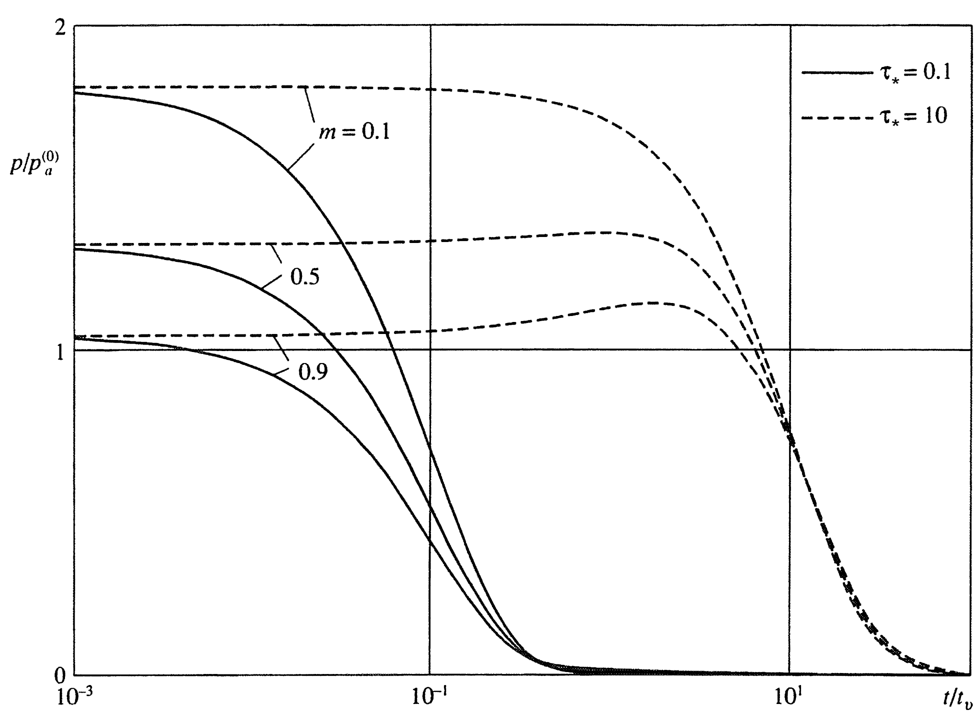

Fig. 1.

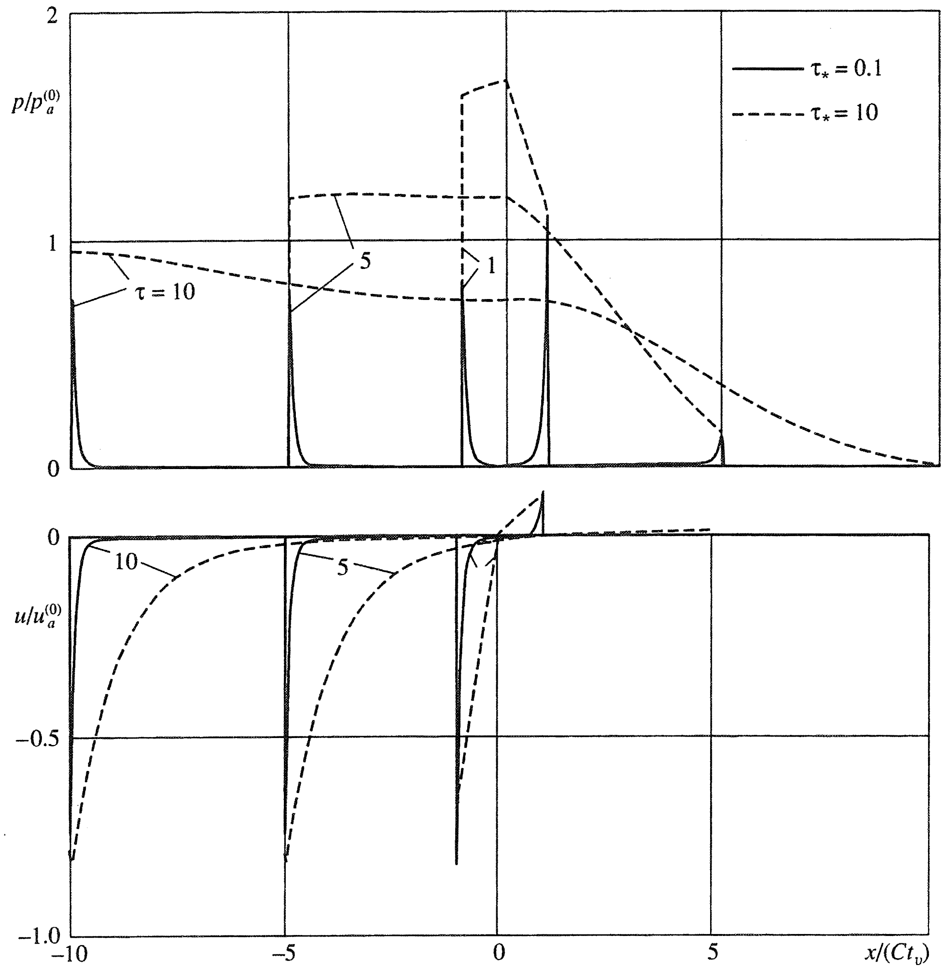
2
$p/p_a^{(0)}$
1
0
$\tau_* = 0.1$
$\tau_* = 10$
5
1
$\tau = 10$
0
$u/u_a^{(0)}$
−0.5
−1.0
10
5
−10
−5
0
5
$x/(Ct_v)$

Fig. 2.

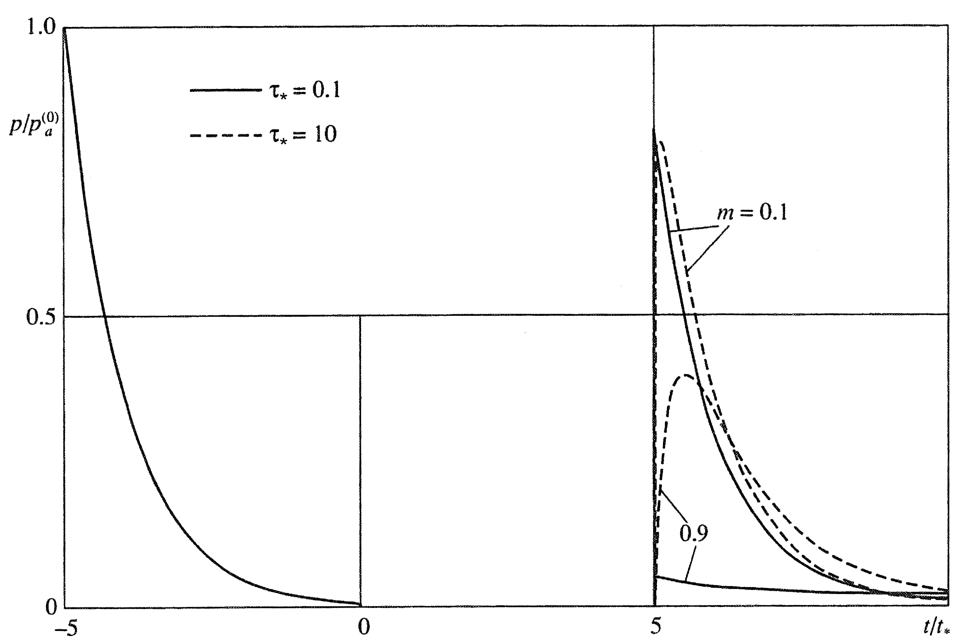
1.0
$p/p_a^{(0)}$
0.5
0
$\tau_* = 0.1$
$\tau_* = 10$
$m = 0.1$
0.9
−5
0
5
t/t_*

Fig. 3.

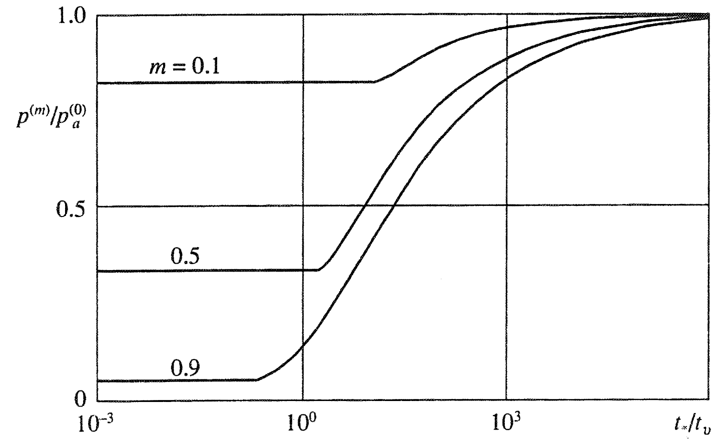

Fig. 4.

In Fig. 3 we illustrate the transformation of the pressure pulse when it is reflected from the porous medium. The calculated oscillograms shown correspond to the readings of a pressure sensor, placed in the zone of pure liquid at the point with dimensionless coordinate $x/(Ct_*) = -5$. The pulse signal shown for $t < 0$ corresponds to the initial incident wave, while the pulse for $t > 0$ corresponds to the reflection of this signal. It can be seen that for shorter signals this "echo" is qualitatively similar to the initial signal and is an instantaneous jump with a "tail" which relaxes to zero, when the amplitude of the leading edge of the jump is equal to the quantity $(1 - m)/(1 + m)$. Hence, the "echo" from short incident signals contains information mainly on the value of the porosity of the skeleton. In the case of long signals (when the value of τ_* lies in the range from 10 to 10^3) the relaxation will be non-monotonic. As calculations show, when the porosity decreases there is a tendency for the formation of intermediate maxima when longer signals are reflected. The position of a maximum for this "echo" from long signals for a fixed value of the porosity is determined by the characteristic time t_υ.

In Fig. 4 we show the envelopes of the dimensionless amplitudes of the reflected signals $p^{(m)}/p_a^{(0)}$ as a function of the dimensionless time τ_*. Hence, one can judge the value of t_υ from the position of a maximum in the oscillogram of the reflected signal and its value. If, moreover, the properties of the saturating liquid, defined by the values of μ and C, are known, the value of the permeability of this skeleton can be determined from the position of a maximum. Consequently, the solutions obtained and calculations using them can be employed to develop methods for an express analysis to determine the porosity and permeability of solid porous materials using modulated pressure waves in the form of a "instantaneous jump and an exponential relaxation zone".

References

1. Maksimov GA, Radchenko AV. Modelling of the intensification of petroleum extraction in acoustic action on a stratum from a well. *Akust Zh* 2005:51. Appendix: 118–32.
2. Nikolayevskii VN, Stepanov GS. Non-linear seismic and acoustic action on the petroleum yield of a stratum. *Akust Zh* 2005:51. Appendix: 150–59.
3. Shagapov VSh, Sultanov ASh, Urmancheyev SF. Solution of the problem of the reflection of linear waves in a liquid from a porous half-space saturated with this liquid. *Zh Prikl Mekh Tekh Fiz* 2006;**47**(5):16–26.
4. Barenblatt GI, Entov VM, Ryzhik VM. *The Motion of Liquids and Gases in Porous Strata*. Moscow: Nedra; 1984.
5. Lavrent'ev MA, Shabat BV. *Methods of the Theory of Functions of a Complex Variable*. Moscow: Nauka; 1987.
6. Carslaw HS, Jaeger JC. *Operational Methods in Applied Mathematics*. Oxford: University Press; 1945.
7. Krylov VI, Shul'gina LT. *Reference Book on Numerical Integration*. Moscow: Nauka; 1966.

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